Relaxation Oscillations

e.) Relaxation Oscillations $\rightarrow$ { Dissipative Nonlinearity / Van-der-Pol's Eqn.

$\rightarrow$ Till now, have considered:

$\Rightarrow$ conservative or 'nearly-conservative' systems (i.e. small, linear dissipation, to resolve resonance)

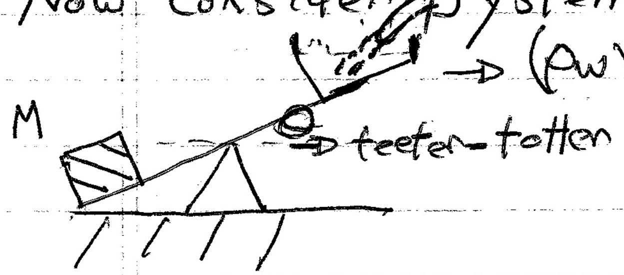
$\Rightarrow$ non-dissipative nonlinearity, only i.e. ϵx^3

$\Rightarrow$ periodic phase space orbits, only i.e. Duffing eqn.

spighot

Now consider systems ala':

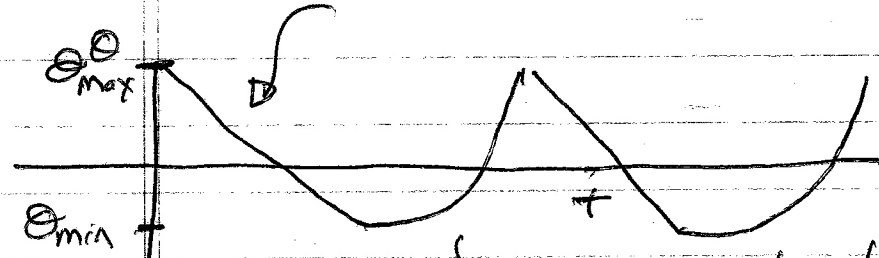
$\rightarrow (\rho_w V_{max} \gg M)$



What happens?:
starting from empty;
① $\rightarrow$ tank fills, Θ decreases (slowly, depends on fill rate)
② $\rightarrow$ $\Theta \rightarrow \Theta_{min}$ $\Rightarrow$ tank empties, quickly!
③ $\rightarrow$ teeter-totter returns to Θ_{max} $(\sim f^{>0})$ (quickly)
and cycle repeats.

thus can graphically depict $\Theta(t)$: ala':

exaggerated slope → set by $\begin{cases} \text{inflow rate} \\ \text{volume } H_2O \text{ at balance} \end{cases}$



→ acceleration ~ g

Observe:

i.e. → sawtooth-like structure → i.e. repeated 'bursts' followed by slower relaxation

→ 2 time scales → slow fill-up and fall
→ quick recovery

H_2O (losses) cmpt.

→ "dissipation" essential. Here "dissipation" is spilling of water at $\Theta = \Theta_{min}$.

→ cyclic behavior

⇒ a classic example of a relaxation-oscillation

A classic mathematical example: the Van-der-Pol Oscillator:

$$\ddot{x} + \nu(x^2-1)\dot{x} + x = 0$$

(self-sustained oscillation of triode)

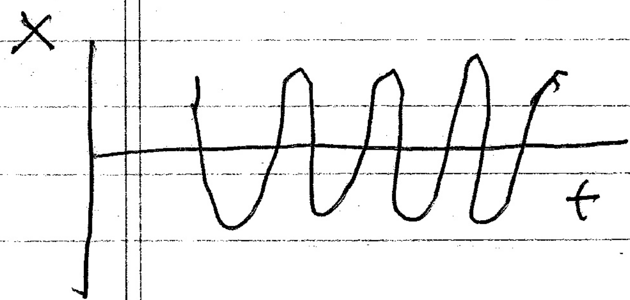
↓ friction — nonlinear; +, − depending on $x \gtrless 1$

Key point: $\lambda_{eff} = \nu(x^2-1)$ ⇒ +, increases for $x^2 > 1$; −, $x^2 < 1$

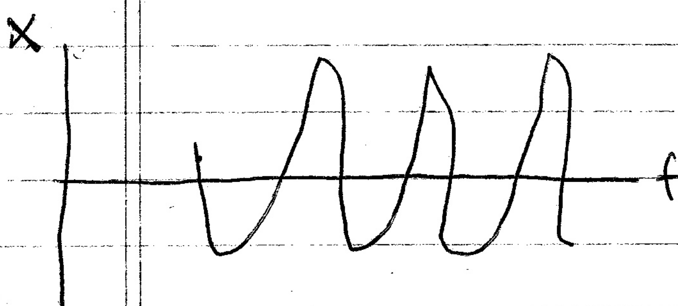
⇒ ⊖ and ⊕ dissipation ⇒ instability and stability

⇒ self-sustaining, self-regulating cycle

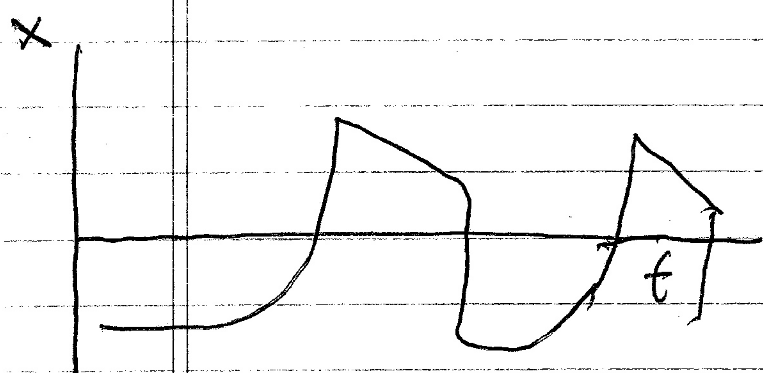
→ $x(t)$ for $\nu = .1,\ 1.0,\ 10$



$\nu = .1$



1.0



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i.e. cyclic, two-time scale phenomena, a la' teeter-totter with H_2O.

∴ study van-der-Pol equation as prototype case (analogous stature to Duffing eqn.) of relaxation oscillator.

→ Some observations

- often $\left\{\begin{array}{l}\text{useful}\\ \text{convenient}\end{array}\right.$ to analyze NL ode's in phase plane ⇒ convert 2^{nd} order eqn. to $\underline{2}$ first order

$\underline{\underline{i.e.}}$ $\ddot{x} + r(x^2-1)\dot{x} + x = 0$ (autonomous!)

$$\left\{\begin{array}{l}\dot{x} = v\\ \dot{v} = -x - r(x^2-1)v\end{array}\right.$$ replace 2^{nd} order NL ode with 2 first-order odes

this is specific instance of:

$$\dot{x} = F(x,y)$$
$$\dot{y} = G(x,y)$$

general form:
In general $\nabla_\Gamma \cdot \underline{V}_\Gamma = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \neq 0$
$\underline{i.e.}$ $\underline{not}$ Hamiltonian

Now, convenient to $\underline{\text{convert}}$ to $\underline{\text{polar}}$ coordinates

$$x^2 + y^2 = r^2 \qquad x = r\cos\theta$$
$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt} \qquad y = r\sin\theta$$

work in phase plane

$\underline{\text{plug in}}$

$$\therefore \frac{dr}{dt} = \cos\theta\, F(r\cos\theta, r\sin\theta) + \sin\theta\, G(r\cos\theta, r\sin\theta)$$

similarly,
$$\frac{d}{dt}(r\cos\theta) = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$$

$$\frac{d}{dt}(r\sin\theta) = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$$

so $x\dot{y} - y\dot{x} =$

$$= \cancel{r\dot{r}\sin\theta\cos\theta} + r^2\dot{\theta}\cos^2\theta - \cancel{r\dot{r}\sin\theta\cos\theta} + r^2\dot{\theta}\sin^2\theta$$

$$= r^2\dot{\theta}$$

$$\Rightarrow \quad \dot{\theta} = r^{-1}\cos\theta\, G(r\cos\theta, r\sin\theta) - r^{-1}\sin\theta\, F(r\cos\theta, r\sin\theta)$$

$\therefore$ finally:

$$\left\{\begin{array}{ll} \dot{r} = \cos\theta\, F + \sin\theta\, G & \dot{x} = F \\ \dot{\theta} = r^{-1}\cos\theta\, G - r^{-1}\sin\theta\, F & \dot{y} = G \end{array}\right\} \quad (*)$$

go to 39 ⇒ simpler example

Now, Van-der-Pol eqn. ⇒

$$\begin{array}{l} \dot{x} = v \\ \dot{v} = -x - r(x^2-1)v \end{array} \Rightarrow \left\{\begin{array}{l} \dot{x} = y \\ \dot{y} = -x - r(x^2-1)y \end{array}\right.$$

$\Rightarrow$ from box on 37 *

$$\dot{r} = r\cos\theta\sin\theta + \sin\theta\left(-r\cos\theta - \gamma(r^2\cos^2\theta - 1)r\sin\theta\right)$$

$$= \cancel{r\cos\theta\sin\theta} - \cancel{r\sin\theta\cos\theta} - \gamma r\sin^2\theta\,(r^2\cos^2\theta - 1)$$

Similarly,

$$\dot{\theta} = r^{-1}\cos\theta\; G - r^{-1}\sin\theta\; F$$

$$= r^{-1}\cos\theta\left(-r\cos\theta - \gamma r\sin\theta\,(r^2\cos^2\theta - 1)\right)$$

$$- r^{-1}\sin\theta\,(r\sin\theta)$$

$$= -1 - \gamma(r^2\cos^2\theta - 1)\cos\theta\sin\theta$$

taking $\gamma \to \epsilon$ for clarity, have:

$$\boxed{\begin{aligned} \dot{r} &= -\epsilon(r^2\cos^2\theta - 1)\,r\sin^2\theta \\ \dot{\theta} &= -1 - \epsilon(r^2\cos^2\theta - 1)\cos\theta\sin\theta \end{aligned}}$$

Phase-plane Eqns. for Van-der-Pol. Osc.

Aside: Phase Plane Portraits – Simpler System

Consider dissipative system:

$$\dot{x} = -y - x(x^2+y^2-1)$$

$$\dot{y} = x - y(x^2+y^2-1)$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\Rightarrow \left.\begin{aligned} \frac{dr}{dt} &= r(1-r^2) \\ \frac{d\theta}{dt} &= 1 \end{aligned}\right\} \text{ phase plane equations}$$

Fixed points: $r=0$
$r=1$

Stability $r = r_0 + \delta r \Rightarrow$

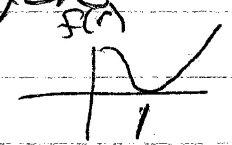
$$\frac{d\,\delta r}{dt} = (r_0+\delta r)\left(1-(r_0+\delta r)^2\right) - r_0(1-r_0^2)$$

$$= \delta r \quad , \; r_0 = 0$$

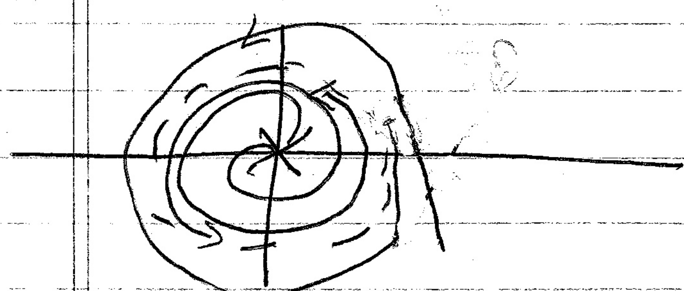
$$\underset{r_0=1}{=} 1(-2\delta r) + \delta r(\cancel{1}-1) = -2\delta r$$

Thus $r_0 = 0$ fixed pt. unstable

$$\left.\begin{array}{l} r=1 \\ \theta = \theta_0 + t \end{array}\right\} \text{ stable limit cycle}$$



i.e.



$r>1 \Rightarrow$ spiral in to $r=1$

$r<1 \Rightarrow$ spiral out to $r=1$

($r=0$ unstable)

(counter-clockwise rotation)

Now, above example particularly simple, since r, θ equations decouple. In real problems, they don't $\Rightarrow$ a la' Van-der-Pol

$$\begin{cases} \dot{r} = -\epsilon(r^2\cos^2\theta - 1)\sin^2\theta = \epsilon F(r, \theta) \\ \dot{\theta} = -1 - \epsilon(r^2\cos^2\theta - 1)\cos\theta\sin\theta = -1 + \epsilon G(r, \theta) \end{cases}$$

$\Rightarrow$ coupled evolution.

For small ϵ, can exploit fact that:

$$\left.\begin{array}{l} \dot{\theta} = -1 + O(\epsilon) \\ \dot{r} = O(\epsilon) \end{array}\right\} \Rightarrow \begin{array}{l} r \text{ nearly } (\epsilon) \text{ constant} \\ \theta \text{ nearly } \theta_0 - t \ (\epsilon) \end{array}$$

nearly constant rotation frequency

∴ expect deviations from circular sinusoidal orbit to accumulate on long time scale (i.e. $t \sim \epsilon^{-1}$).

⇒ Method of Averaging → a type of multiple time scale P.T. (yet another)

A Trivial Example: $\ddot{x} + 2\epsilon\dot{x} + x = 0$

Exact: $x = a_0 e^{-i\omega t} \Rightarrow -\omega^2 + (-2\epsilon i\omega) + 1 = 0$

∴ $\omega = \omega_r + i\omega_i \Rightarrow$

$$-(\omega_r^2 - \omega_i^2 + 2i\omega_r\omega_i) - 2\epsilon(i\omega_r - \omega_i) + 1 = 0$$

$$i\left[-2\omega_r\omega_i - 2\epsilon\omega_r\right] = 0$$

$$-(\omega_r^2 - \omega_i^2) + 1 - 2\epsilon\omega_i = 0$$

∴ $\omega_i = -\epsilon$

$$\Rightarrow -(\omega_r^2 - \epsilon^2) + 1 - 2\epsilon^2 = 0$$

$$\omega_r^2 = 1 - \epsilon^2$$

∴ $x = a_0 e^{-\epsilon t} \cos\left[(1-\epsilon^2)^{1/2} t\right]$

~~de~~ clearly oscillates at $\omega \sim 1$, with slow decay on $\tau \sim 1/\epsilon$

ala' averaging:

– r, θ equations

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -x - 2\epsilon v$$
$$\equiv F \qquad\qquad \equiv G$$

as above $\Rightarrow$

$$\frac{dr}{dt} = -2\epsilon r \sin^2\theta \quad , \quad \frac{d\theta}{dt} = -1 - \epsilon \sin 2\theta$$

Now, as interested in long-time, slowly-building-up cumulative effects, average equations over typical circuit

i.e. $a(t) = \oint ds \, r(t+s) / \oint ds$ $s \equiv$ fast circuit variable.

avg. amplitude (circuit), so

(N.B.: Note method of avging. is inherently a two-time-scale method, i.e.)

– $dr/dt = O(\epsilon)$, $d\theta/dt = \omega + O(\epsilon)$

slow   fast $O(1)$

then $\begin{cases} \text{fast time scale} = \omega^{-1} \\ \text{slow time scale} = \left(\frac{1}{r}\frac{dr}{dt}\right)^{-1} \end{cases}$

and point of averaging is to eliminate the fast time scale!

<u>so</u> $$a(t) = \int_{2\pi}^{0} r(t+s)\,ds \Big/ \int_{2\pi}^{0} ds$$

$$\int_{2\pi}^{0} ds \qquad \frac{d\theta}{dt} = -1 + o(\epsilon)$$

$$= \int_{2\pi}^{0} r(t+s)\,\frac{d\theta}{d\theta/ds} \Big/ \int_{2\pi}^{0} \frac{d\theta}{d\theta/ds}$$

<u>but</u> $\frac{d\theta}{ds} = -1 + o(\epsilon)$

$\Rightarrow$ $$a(t) = \frac{1}{2\pi}\int_{0}^{2\pi} r(t+s)\,d\theta + o(\epsilon)$$

then $$\frac{da}{dt} = \frac{d}{dt}\bar{r} = \frac{(1+o(\epsilon))}{2\pi}\int_{0}^{2\pi} \frac{dr(t+s)}{ds}\,d\theta$$

deriv. on <u>slow</u> scale

and orbit eqns $\Rightarrow$

$$= \frac{(1+o(\epsilon))}{2\pi}\int_{0}^{2\pi} (-2\epsilon r \sin^2\theta)\,d\theta$$

$d\theta/ds$ factor

$$= \frac{(1+o(\epsilon))}{2\pi}\int_{0}^{2\pi} (-2\epsilon r)\left(\frac{1}{2} - \frac{1}{2}\sin 2\theta\right) d\theta$$

$$\Rightarrow \quad \frac{da}{dt} = -\epsilon a \quad + O(\epsilon^2)$$

Similarly;

$$\omega(t) = \oint \frac{d\theta}{ds}(t+s)\, ds \Big/ \oint ds$$

slow variation in ω

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{ds}(t+s)\, d\theta$$

but $\frac{d\theta}{dt} = -1 - \epsilon \sin 2\theta$

$$\omega(t) = \frac{1}{2\pi} \int_0^{2\pi} \left(-1 - \epsilon \sin 2\theta\right) d\theta \quad + O(\epsilon^2)$$

$$= -1 \quad + O(\epsilon^2)$$

i.e. $\left\{ \begin{array}{l} O(\epsilon) \text{ correction} \\ \text{vanishes} \end{array} \right.$

so
$$\boxed{\begin{array}{l} \frac{da}{dt} = -\epsilon a \\ \frac{d\theta}{dt} = \omega(t) = 1 \end{array}}$$

$\rightarrow$ avgd. eqns. to leading order

In particular 'natural' frequency of $\omega = -1$ with decaying ampl.

so/ to leading order:

$$\left.\begin{aligned}\frac{da}{dt} &= -\epsilon a \\ \omega(t) &= -1\end{aligned}\right\} \Rightarrow \begin{aligned} a(t) &= a_0 e^{-\epsilon t} \\ \theta &= (t_0 - t)\end{aligned}$$

so $x \cong a_0 e^{-\epsilon t} \cos(t - t_0)$

but exact: $x = a_0 \underbrace{e^{-\epsilon t}}_{\text{slow scale}} \cos\underbrace{\left((1-\epsilon^2)^{1/2}(t-t_0)\right)}_{\text{fast scale}}$

agree to $O(\epsilon)$!

Now, proceed with Van der Pol equation, for which:

$$\frac{dr}{dt} = -\epsilon\,(r^2\cos^2\theta - 1)\, r \sin^2\theta$$

$$\frac{d\theta}{dt} = -1 - \epsilon\,(r^2\cos^2\theta - 1)\cos\theta \sin\theta$$

Note same generic structure as example

$dr/dt = O(\epsilon)$

$d\theta/dt = -1 + O(\epsilon)$

Thus, can immediately write:

$$\frac{da}{dt} = \frac{-\epsilon}{2\pi}\oint_0^{2\pi}(r^2\cos^2\theta - 1)\, r\sin^2\theta \, d\theta + O(\epsilon^2)$$

$2\cos\theta\sin\theta = \sin 2\theta$

$$= \frac{-\epsilon}{2\pi}\oint_0^{2\pi} r\left[(r^2/4)\sin^2 2\theta - 1/2(1-\cos 2\theta)\right] d\theta$$

$\Rightarrow$

$$\frac{da}{dt} = \frac{\cancel{4}}{2}\epsilon\, a(1 - 1/4\, a^2) + O(\epsilon^2)$$

Similarly,

$$\omega + 1 = -\frac{\epsilon}{2\pi}\int_0^{2\pi}(r^2\cos^2\theta - 1)\cos\theta\sin\theta\, d\theta$$

$$= O(\epsilon^2)$$

$$\omega = -1$$

$\Rightarrow$ averaged equations:

$$\boxed{\frac{da}{dt} = \frac{\epsilon a}{2}(1 - a^2/4) \qquad \omega = -1}$$

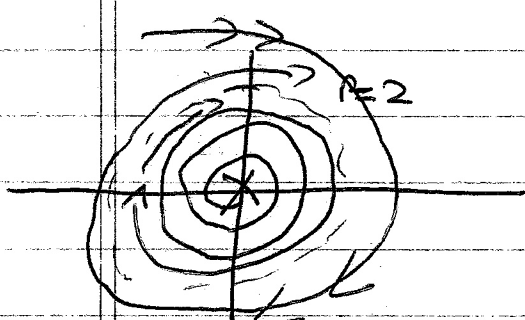
"Natural frequency" of VdP oscillator is $\omega = -1$. Non-trivial attractor at $a = 2$

so $x = 2\cos(\theta + t)$
$y = 2\sin(\theta + t)$

Now, equation for $a(t)$ similar to earlier example i.e.

equilibria at $a=0 \rightarrow$ unstable focus

$a=2 \rightarrow$ stable limit cycle ($\omega=-1 \Rightarrow$ clockwise rotation of cycle)



$r>2 \rightarrow$ spiral in

$r<2 \rightarrow$ spiral out

→ But what happens when ϵ isn't "small"? → i.e. { sawtooth vs. sinusoid

→ averaging recovers ~sinusoid

To answer this, more convenient to consider Rayleigh equation, instead Van-der-Pol eqn.

i.e. Rayleigh equation;

$$\ddot{x} + \epsilon\left[\frac{\dot{x}^3}{3} - \dot{x}\right] + x = 0 \quad \Rightarrow$$

↳ +, − NL dissipation

N.B. if d/dt thru:

$$\dddot{x} + \epsilon\left[\dot{x}^2 - 1\right]\ddot{x} + \dot{x} = 0$$

and $y = \dot{x}$ ⇒

$$\ddot{y} + \epsilon\left[y^2 - 1\right]\dot{y} + y = 0$$

recovers Van-der-Pol eqn. ⇔ related systems

Now, in phase plane;

R.E. ⇒ $$\frac{d^2u}{dt^2} + \epsilon\left\{\frac{1}{3}\left(\frac{du}{dt}\right)^3 - \frac{du}{dt}\right\} + u = 0$$

so $$\frac{du}{dt} = x \quad ; \quad \frac{dx}{dt} = -u - \epsilon\left(\frac{x^3}{3} - x\right)$$

Further; $w \equiv u/\epsilon$

$$F(x) = x^3/3 - x$$

$$\Rightarrow \frac{dx}{dw} = \frac{dx/dt}{dw/dt} = \epsilon \frac{dx/dt}{du/dt}$$ (yields orbits in x, w plane)

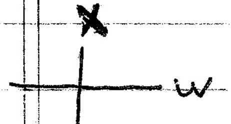
but $$\frac{dx}{dt} = -\epsilon \frac{u}{\epsilon} - \epsilon\left(\frac{1}{3}x^3 - x\right)$$

$$= -\epsilon\left(w + F(x)\right)$$

$$\Rightarrow \boxed{\frac{dx}{dw} = -\epsilon^2 \frac{w + F(x)}{x}}$$

Thus, can immediately observe:

- as $\epsilon \to \infty$, $dx/dw \underset{(\text{sign})}{\to \infty}$ $\to \epsilon$! except on $w + F(x) = 0$
- $dx/dw = 0$ on $w + F(x) = 0$, $\forall \epsilon$

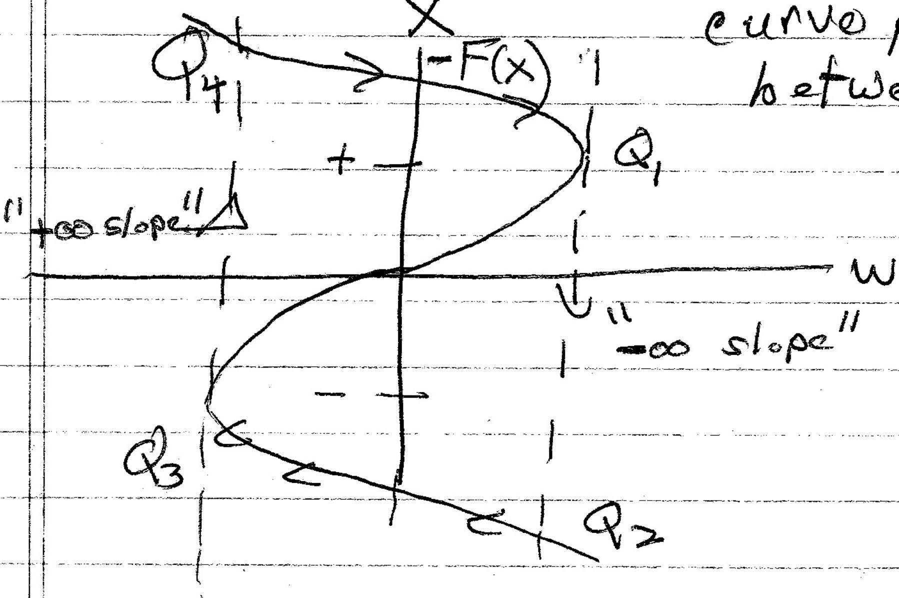
$\therefore$ 

- in (w, x) plane, orbit parallels x-axis except where $w + F(x) = 0$. Indeed,
(but $\pm\infty$ slope!)

dx/dw large, > 0 $\quad (w + F(x))/x < 0$
" " large, < 0 $\quad (w + F(x))/x > 0$

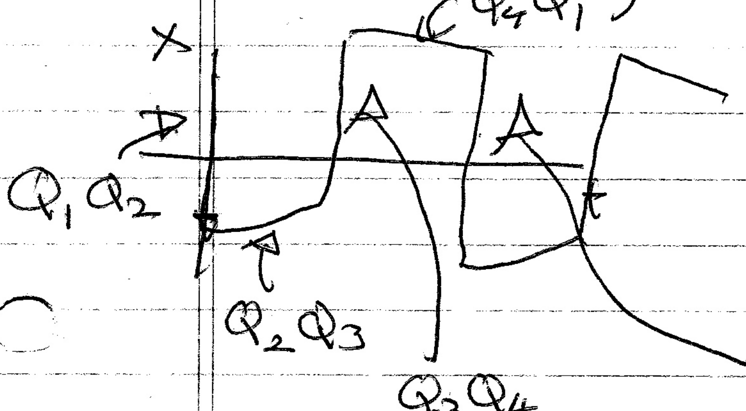
$\therefore$ orbit starting at $t = 0$ moves vertically toward curve $w = -F(x)$, and then describes closed orbit in w, x plane, along curve pts (with jumps between)

i.e.



- $Q_1 \to Q_2 \to Q_3 \to Q_4 \to Q_1$ is closed limit cycle in x, w plane.

- if plot $x(t)$, can recover relaxation-oscillation sawtooth.



{ 2 bifurcation jumps
{ 2 regions slow evolution

etc.

- note relaxation oscillation intimately connected to bi-stability — reason for Rayleigh choice